Measuring the leverage effect in a high frequency trading framework

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Abstract

Multi-factor stochastic volatility models of the financial time series can have important applications in portfolio management and pricing/hedging of financial instruments. Based on the semi-martingale paradigm, we focus on the study and the estimation of the leverage effect, defined as the covariance between the price and the volatility process and modeled as a stochastic process. Our estimation procedure is based only on a pre-estimation of the Fourier coefficients of the volatility process. This approach constitutes a novelty in comparison with the non-parametric leverage estimators proposed in the literature, generally based on a pre-estimation of the spot volatility, and it can be directly applied to estimate the leverage effect in the case of irregular trading observations and in the presence of microstructure noise contaminations, i.e. in a high frequency framework. The finite sample performances of the Fourier estimator of the leverage are tested in numerical simulations and in an empirical application to the S&P 500 index futures.

1 Introduction

The models used to describe the dynamics of the financial time series have to incorporate the speed and complexity of the modern financial markets. Since 1999 after the U.S. Securities and Exchange Commission (SEC) authorized electronic exchanges, the high frequency trading accounts approximatively for 50% of all trading volume just taking into account the US equity market[1]. Nowadays, technological progress along with the growing dominance of electronic trading allow to record market activity with high precision leading to advanced and comprehensive data sets. The historical data analysis of the financial time series, therefore, cannot avoid the use of the aforementioned data as long as their underlying models have to show a richer structure in the price/volatility dynamics in order to fit the features of data with time aggregation of minutes and seconds. In this direction, a first fundamental step is reinterpreting the classical stylized facts of the financial time series in order to improve our understanding of the matters concerned.

The *leverage effect* is one of the classical stylized facts observed in the security return distributions, along with the well-known fat tails, skewness and heteroscedasticity, and it is closely related to the stochastic

[1] *Equity Market Structure Literature Review, Part II: High Frequency Trading*, Staff of the Division of Trading and Markets (U.S. Securities and Exchange Commission), March 18, 2014.
Market microstructure confronting many view points, F. Abergel, J.P. Bouchaud, T. Foucault, C.A. Lehalle, M.Rosenbaum, 2012.

nature of the volatility dynamics. It refers to the relationship between returns and their corresponding volatilities which tends to be negatively correlated. One possible economic interpretation of this phenomenon was developed in Black (1976) and Christie (1982) and it is connected with the concept of *financial leverage* (debt-to-equity ratio). As asset prices decline, companies become automatically more leveraged since the relative value of their debts rises relative to that of their equities. The probability of default rises and then their stocks become riskier, hence more volatile. As discussed in Ait-Sahalia, Fan and Li (2013) being the most prevalent economic interpretation in literature, the name *leverage* is also used to describe the statistical correlation between the prices and their corresponding volatilities.

In order to capture the leverage effect in modeling terms, a classical approach consists in using a constant correlation structure between the price and its corresponding volatility - e.g. Heston (1993) and Barndorff-Nielsen and Shepard (2002). Recent empirical works, however, emphasized that this effect is not constant, but itself evolves in time- see Yu (2005) among others- and that there may be important asymmetries in the way in which the volatility responds to price changes as studied in Carr and Wu (2007), Bandi and Renò (2012) - e.g. in presence of positive shocks (positive returns) the volatility may not change or even change positively. These findings motivate the growth of sophisticated models like the class of the multi-factor stochastic volatility models. The correlation structure between price and volatility can be modeled as a state space dependent variable or as in Veraart and Veraart (2012) as a stochastic process itself.

Generally, calibrating these models to market information is rather complicated because estimation procedures of the leverage and of the variance of the volatility processes have not been extensively studied under general hypotheses and an inference on these models cannot avoid estimations of these quantities. We will introduce a non-parametric procedure for the leverage estimation based on the Fourier analysis developed in Malliavin and Mancino (2002, 2009), showing the versatility of this estimation procedure and its effectiveness when dealing with high frequency data. The Fourier methodology has already been applied in estimating second order latent quantity as the variance of the volatility in Curato, Mancino and Sanfelici (2014).

We assume that the underlying dynamics of the price and volatility processes are governed by two continuous semi-martingales, correlated by means of a stochastic process $\rho(t)$. We do not assume any specific functional form for the volatility, for the variance of the volatility and for the correlation processes between the Brownian motions driving the price and volatility. In particular, the Heston model and the Generalized Heston model proposed in Veraart and Veraart (2012) are included in our framework. With respect to the other non-parametric estimators present in literature that involved the use of high frequency data- Barucci and Mancino (2010), Cuchiero and Teichmann (2013), Bandi and Renò (2012), Mykland and Wang (2014) - we define integrated and spot estimations of the leverage in a novel way i.e. by using only a pre-estimation of the Fourier coefficients of the *latent* volatility process. In Barucci and Mancino (2010), Cuchiero and Teichmann (2013) two different non parametric procedures with several features in common are presented. First, the authors estimate the spot volatility process using respectively the Fourier estimator developed in Malliavin and Mancino (2009) and a Fourier estimator constructed starting by a jump-robust estimation of the covariance process. Secondly, they estimate the leverage function using the estimated spot volatility instead of its unknown paths. Bandi and Renò (2012) and Mykland and Wang (2014) develop non parametric procedures suitable for *local* stochastic leverage models. Also these methods are based on a pre-estimation of the spot volatility function since the leverage is defined as a state space dependent function of the volatility process. These estimators, however, do not take into account the microstructure contamination effects that might appear when dealing with data having time aggregation less than five minutes- Hautsch (2012). These effects might spoil the estimation process, as the spot volatility estimators are quite sensitive to noise. Avoiding the estimation of the spot volatility allows us to define consistent estimators that, without any manipulation of the data, are robust under microstructure noise and irregular trading (unevenly observations of the price path).

We investigate the robustness to microstructure effects of the integrated estimator via numerical simulations. We generate two data-sets by means of an Euler-discretization of the Heston and the Generalized Heston model and we study the performances of the leverage estimator in different scenarios. The simulation results corroborate the theoretical results and also show the features of the Fourier methodology in realistic frameworks. We are able to construct efficient estimations of the integrated leverage and to conduct an analysis on the sensitivity to the choice of the cutting parameters. An empirical application to S&P 500 index futures is also presented.

The chapter is organized as follows. The model setting is carefully described in Section 2. In Section 3 we define the Fourier estimators of the spot and integrated leverage and prove their consistency. Finally, in Section 4 the Monte-Carlo and empirical results are shown. Section 5 concludes.

2 Model Setting

We consider the log-price and the volatility processes defined on a probability space $(\Omega,(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$ satisfying the usual conditions and following the It$\hat{o}$ stochastic differential equations

$$\begin{cases} dp(t) = \sigma(t)dW(t) + a(t)dt \\ dv(t) = \gamma(t)dZ(t) + b(t)dt, \end{cases} \tag{1}$$

where $v(t)=\sigma^2(t)$ and W(t) and Z(t) are correlated Brownian motions. The correlation process between the Brownian motions is defined as

$$<dW(t),dZ(t)>= \rho(t)dt,$$

where the brackets stands for the It$\hat{o}$ contraction and $\rho(t)$ is a process with value in [-1,1].

A standard no-arbitrage condition suggests that the security price must be a semimartingale as prescribed by Back (1991) and Delbaen and Schachermayer (1994). These types of processes obey to the fundamental theorem of asset pricing and, as a result, are used extensively in financial econometrics, (see Ghysels, Harvey and Renault (1996) for a review). We think of this model as the model governing an underlying efficient price process, i.e. the price that would be observed in the absence of market frictions. We do not assume any specific functional form of the volatility, of the volatility of volatility and of the correlation processes, thus we are working in a model free setting. In particular, such parametric models as Heston, CEV, and the Generalized Heston Model defined in Veraart and Veraart (2012) fit our assumptions.

Scaling the unit of time, we can always reduce ourselves to the case in which the time window [0,T] becomes [0,2π]. For this reason, in what follows we will consider the time window to be [0,2π], which is the most suitable choice if we want to apply the Fourier analysis. We make the following hypotheses on the processes that appear in (1):

- **H.1** a(t), b(t), σ(t), γ(t), ρ(t) are continuous in [0,2π] and adapted to the filtration $\mathcal{F}_t$ with values in $\mathbb{R}$,
- **H.2** $\forall$ p $\geq$ 1

$$E\left[\sup_{t\in[0,2\pi]}|a(t)|^p\right]<\infty,\quad E\left[\sup_{t\in[0,2\pi]}|b(t)|^p\right]<\infty,$$

$$E\left[\sup_{t\in[0,2\pi]}|\sigma(t)|^p\right]<\infty,\quad E\left[\sup_{t\in[0,2\pi]}|\gamma(t)|^p\right]<\infty,$$

$$E\left[\sup_{t\in[0,2\pi]}|\rho(t)|^p\right]<\infty,$$

- **H.3** $\forall$ p ≥ 1, the processes a(t), b(t), σ(t), γ(t) $\in\ \mathbb{D}^{1,p}$ and

$$E\left[\sup_{s,t\in[0,2\pi]}|\mathcal{D}_s a(t)|^p\right]<\infty,\quad E\left[\sup_{s,t\in[0,2\pi]}|\mathcal{D}_s b(t)|^p\right]<\infty,$$

$$E\left[\sup_{s,t\in[0,2\pi]}|\mathcal{D}_s \sigma(t)|^p\right]<\infty,\quad E\left[\sup_{s,t\in[0,2\pi]}|\mathcal{D}_s \gamma(t)|^p\right]<\infty,$$

  where $\mathbb{D}^{1,p}$ is the Sobolev space of the generalized derivative in the sense of Malliavin and $\mathcal{D}$ stands for the Malliavin derivative, Nualart (2006).

We are interested in estimating the *spot leverage process* $\eta(t)$, which is defined by means of the Itô contraction between the price and volatility as

$$<dp(t),dv(t)>=\eta(t)dt, \tag{2}$$

and the *integrated* quantity

$$\eta^{[1]}=\int_0^{2\pi}\eta(t)\,dt. \tag{3}$$

3 Computation of the leverage using the Fourier methodology

The leverage process represents the covariance between the price and the volatility process as stated in (2). The Fourier methodology developed by Malliavin and Mancino (2002,2009) for the estimation of the covariance between asset returns can be adapted to get estimations also in this context. Before proceeding, we recall some definitions from harmonic analysis theory, see e.g. Malliavin (1995).

Given ϕ defined on the Hilbert space $L_2([0,2\pi])$ of the complex valued functions, we consider its *Fourier coefficients*, defined on the group of the integers ℤ by the formula

$$c_h(\phi) := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ht}\ \phi(t)dt \qquad \text{for all h } \in \mathbb{Z}. \tag{4}$$

The set of the Fourier coefficients represents the *coordinates* of ϕ respect to the orthonormal basis $\{e_h\,(t) = \mathrm{e}^{iht}\ \ s.t.\ h\ \in \mathbb{Z}\}$ of the Hilbert space $L_2([0,2\pi])$. Thus, starting by an arbitrary number of Fourier coefficients as $(c_1, \dots, c_N)$, we can reconstruct a trigonometric approximation of ϕ by means of the orthogonal projection of the function onto the space $< e_1, \dots, e_N >$

$$\pi_N\,(\phi) = \sum_{i=1}^{N} e_i\ c_i(\phi),$$

that can be interpreted as an *estimation* of the function ϕ with arbitrary precision, see the works of De La Vallée Poussin (1919), Favard (1937) and Zamansky (1949) for further details. Therefore, *an arbitrary sequence of Fourier coefficients includes the necessary information to get an estimation of* ϕ. Given two functions Φ and Ψon the integers ℤ, we say that the *Bohr convolution product* exists if the following limit exists for all integers h

$$(\Phi * \Psi)(h) := \lim_{N\to\infty} \frac{1}{2N+1}\sum_{|l|\le N} \Phi(l)\Psi(h-l).$$

Then, the following identity relating the Fourier coefficients of dp and dν to the Fourier coefficients of the process η(t) holds

$$c_h(\eta) := \lim_{N\to\infty} \frac{2\pi}{2N+1}\ \sum_{|l|\le N} c_l\,(\mathrm{d}\nu)\ c_{h-l}(dp)\,, \tag{5}$$

where the convergence is attained in probability, see Theorem 2.1 in Malliavin and Mancino (2009). The above identity is feasible only when continuous observations of the price and volatility process are available. Before turning to a more realistic framework, we need the following considerations.

We start from the definition of the approximate Fourier coefficients, obtained in (5) by dropping the limit operator

$$c_h(\eta_N) := \frac{2\pi}{2N+1} \sum_{|l|\leq N} c_l\,(\mathrm{d}\nu)\, c_{h-l}(dp), \tag{6}$$

where the Fourier coefficients

$$c_l(dp) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}lt}\, dp(t), \tag{7}$$

for all $|l| \leq 2N$ and $c_l(d\nu)$ can be defined by means of the integration by part formula for all $|l| \neq 0$ as

$$c_l(d\nu) = \mathrm{i}l c_l(\nu) + \frac{1}{2\pi}\big(\nu(2\pi) - \nu(0)\big). \tag{8}$$

It is evident by (8) that pre-estimating the volatility path is a necessary step in order to define the coefficients (6). This is the methodology followed in Barucci and Mancino (2010).

In the present work, we modify the Bohr convolution product leading to the definition (6) by replacing the coefficients $c_l(d\nu)$ with $\mathrm{i}l c_l(\nu)$ for all $l \neq 0$. Therefore, we propose the following

$$\hat{c}_h(\eta_N) := \frac{2\pi}{2N+1} \sum_{|l|\leq N} \mathrm{i}l\, c_l\,(\nu)\, c_{h-l}(dp), \tag{9}$$

in which only a pre-estimation of the Fourier coefficients of the volatility is required.

We note that the spot volatility enters implicitly in the definition (9) because its Fourier coefficients define a trigonometric approximation of ν(t). The effectiveness of the definition (9) shows when we observe the log-price process at discrete unevenly spaced times. In fact, the instability of the spot volatility estimations at the boundary of a finite sample is a well-known result (end effects). Even the Fourier spot volatility estimators used in Barucci and Mancino (2010) and Cuchiero and Teichmann (2013), that are more suitable to deal with unevenly spaced data, introduces a bias term if evaluated at the boundary of the time window $[0,2\pi]$. The definition (9) overcomes the above problems allowing to define a consistent estimator. We now define the procedure that allows to define the Fourier coefficients of the leverage starting by *discrete observations* of the price process.

- **Step 1:** we start by pre-estimating the Fourier coefficients of the volatility. We assume p(t) is observed at a discrete unevenly spaced grid
  $\mathcal{S}_n := \{0 = t_{0,n} \leq t_{1,n} \leq .. \leq t_{i,n} \leq .. \leq t_{k_n,n} = 2\pi\}$, for all i=0,...,$k_n$ and $k_n \leq n$,
  and we define $\rho(n) := \max_{i=0,\dots,k_n-1} |t_{i+1,n} - t_{i,n}|$ and the discrete observed return as
  $$\delta_{i,n}(p) = p(t_{i+1,n}) - p(t_{i,n}) \text{ for all i=0,...,} k_n\text{-1.}$$

Therefore, by means of the classical definition of the discrete Fourier transform, we estimate $c_s(dp)$ as

$$c_s(dp_n) = \frac{1}{2\pi} \sum_{i=0}^{k_n-1} \mathrm{e}^{-\mathrm{i}st_{i,n}} \ \ \delta_{i,n}(p), \tag{10}$$

for any integer s such that $|s| \leq 2M + N$.
We define the Fourier coefficients estimators of the volatility process as in Malliavin and Mancino (2002)

$$c_l(v_{n,M}) := \frac{2\pi}{2M+1} \sum_{|s| \leq M} c_s(\mathrm{d}p_n)\, c_{l-s}(dp_n) \tag{11}$$

for any integer l such that $|l| \leq 2N$.

- **Step 2:** by means of the definition (9) and the estimations (10) and (11), we get the estimators of the Fourier coefficients of the leverage processes for any integer h such that $|h| \leq N$

$$\hat{c}_h(\eta_{n,M,N}) := \frac{2\pi}{2N+1} \sum_{|l| \leq N} \mathrm{i}l\; c_l\left(v_{n,M}\right) c_{h-l}(dp_n). \tag{12}$$

The consistency of the estimator (12) is proved in the following Theorem.

Theorem 3.1. For all $|h| \leq N$, let $\hat{c}_h(\eta_{n,M,N})$ be the h-th Fourier coefficient estimator of the leverage process defined in (12). We assume that the hypotheses (**H**) and

$$\frac{N^2}{M} \to 0 \ \text{ and } \ M\rho(n) \to a \tag{13}$$

with $a \in \left(0, \frac{1}{2}\right)$, hold true as $n, N, M \ \to \infty$ and $\rho(n) \to 0$. Then

$$\hat{c}_h(\eta_{n,M,N}) \overset{\mathbb{P}}{\to} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ht}\, \eta(t) \ dt. \tag{14}$$

Remark 3.2. The range prescribed for the parameter *a* is connected with the Nyquist frequency.
In this context, the cutting frequencies *M* and *N* that denote respectively the number of the Fourier coefficients of the return and of the volatility process to use in the definition (12) have an order of magnitude less than n/2- the so called Nyquist frequency- in order to avoid aliasing effects.

Proof (Theorem 3.1.):
Hereafter, let us $\phi_n(t) := sup_{i=0,\dots,k_n}\{t_{i,n}: t_{i,n} \leq t\}$, we will refer to the discrete Fourier coefficients of the return process by using the following equivalent integral definition

$$c_s(dp_n) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}s\phi_n(t)}\ dp(t) \tag{15}$$

for all $|s| \leq 2M + N$.
The notation D_M stands for the normalized Dirichlet kernel. In its continuous definition it is defined as

$$D_M(s) = \frac{1}{2M+1} \sum_{|k|\leq M} \mathrm{e}^{-\mathrm{i}ks}, \quad \text{for all} \quad M \in \mathbb{N}$$

and by substituting $\phi_n(s)$ instead of s we will refer to its discrete version.

We can decompose

$$\hat{c}_h(\eta_{n,M,N}) - \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ht}\,\eta(t)\ dt$$

$$= \frac{2\pi}{2N+1}\sum_{|l|\leq N} \mathrm{i}l\ c_l(\nu_{n,M})\ c_{h-l}(dp_n) - \frac{2\pi}{2N+1}\sum_{|l|\leq N} \mathrm{i}l\ c_l(\nu)c_{h-l}(dp) \tag{16}$$

$$+ \frac{2\pi}{2N+1}\sum_{|l|\leq N} \mathrm{i}l\ c_l(\nu)\ c_{h-l}(dp) - \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ht}\,\eta(t)\ dt\ . \tag{17}$$

In what follows, the constant C will denote a suitable constant that may not necessarily be the same.

Applying the Cauchy-Schwartz inequality to (16) we have that in L_1-norm

$$E\left[\left|\frac{2\pi}{2N+1}\sum_{|l|\leq N} \mathrm{i}l\ c_l(\nu_{n,M})\ c_{h-l}(dp_n) - \frac{2\pi}{2N+1}\sum_{|l|\leq N} \mathrm{i}l\ c_l(\nu)c_{h-l}(dp)\right|\right]$$

$$= E\left[\left|\frac{2\pi}{2N+1}\sum_{|l|\leq N} \mathrm{i}l\ c_l(\nu_{n,M})\left(c_{h-l}(dp_n) - c_{h-l}(dp)\right) + \mathrm{i}l\ c_{h-l}(dp)\left(c_l(\nu_{n,M}) - c_l(\nu)\right)\right|\right]$$

$$\leq \frac{2\pi}{2N+1} \sum_{|l|\leq N} |l| \Bigg(E\left[c_l(\nu_{n,M})^2 \right]^{\frac{1}{2}} E\left[\left(c_{h-l}(dp_n) - c_{h-l}(dp) \right)^2 \right]^{\frac{1}{2}} + E[\, c_{h-l}(\mathrm{dp})^2]^{\frac{1}{2}} E\left[\left(c_l(\nu_{n,M}) - c_l(d\nu) \right)^2 \right]^{\frac{1}{2}} \Bigg). \tag{18}$$

The Fourier coefficients of the return process are bounded under the hypotheses (**H**) and

$$E\left[\left(c_l(dp_n) - c_l(dp) \right)^2 \right] \leq E\left[\left(\frac{1}{2\pi} \int_0^{2\pi} \left(\mathrm{e}^{-\mathrm{il}\phi_n(t)} - \mathrm{e}^{-\mathrm{ilt}} \right) \sigma(t) dW(t) + \frac{1}{2\pi} \int_0^{2\pi} \left(\mathrm{e}^{-\mathrm{il}\phi_n(t)} - \mathrm{e}^{-\mathrm{ilt}} \right) a(t) dt \right)^2 \right] \leq C\, N^2\, \rho^2(n) \tag{19}$$

for each $|l| \leq 2N$ after using the It$\hat{o}$ identity and the Taylor's formula.

From the definition (11) and applied the It$\hat{o}$ formula to the product $c_s(\mathrm{d}p_n)\, c_{l-s}(dp_n)$ we obtain the following decomposition regarding the discrete Fourier coefficients of the volatility process

$$c_l\left(\nu_{n,M}\right) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{il}\phi_n(t)} \nu(t)\, dt + I_{M,n} + \tilde{I}_{M,n} + H^1_{M.n} + H^2_{M,n} + H^3_{M,n} + \widetilde{H}^1_{M.n} + \widetilde{H}^2_{M,n} + \widetilde{H}^3_{M,n}$$

where $I_{M,n}$ and $\tilde{I}_{M,n}$ are the contributions due to the diffusion part of dp in (1)

$$I_{M,n} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^t \mathrm{e}^{-\mathrm{il}\phi_n(u)}\, D_M\left(\phi_n(t) - \phi_n(u)\right) \sigma(u) dW(u)\, \sigma(t) d\, W(t)$$

$$\tilde{I}_{M,n} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{il}\phi_n(t)} \int_0^t D_M\left(\phi_n(t) - \phi_n(u)\right) \sigma(u) dW(u)\, \sigma(t) d\, W(t). \tag{20}$$

and the other terms are the contributions of the drift part

$$H^1_{M.n} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^t \mathrm{e}^{-\mathrm{il}\phi_n(u)}\, D_M\left(\phi_n(t) - \phi_n(u)\right) a(u) du\, \sigma(t) d\, W(t) \tag{21}$$

$$H^2_{M.n} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^t \mathrm{e}^{-\mathrm{il}\phi_n(u)}\, D_M\left(\phi_n(t) - \phi_n(u)\right) \sigma(u) dW(u)\, a(t) dt \tag{22}$$

$$H^3_{M.n} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^t \mathrm{e}^{-\mathrm{il}\phi_n(u)}\, D_M\left(\phi_n(t) - \phi_n(u)\right) a(u) dW(u)\, a(t) d\, W(t) \tag{23}$$

and $\widetilde{H}^1_{M.n}$, $\widetilde{H}^2_{M,n}$ and $\widetilde{H}^3_{M,n}$ are defined in a symmetric way.

Using subsequently the Itô identity, the Cauchy-Schwartz and the Burkholder-Gundy inequalities the term (20) becomes in L_2-norm

$$E\left[\left(I_{M,n}\right)^2\right] = E\left[\frac{1}{4\pi}\int_0^{2\pi}\left(\int_0^t \mathrm{e}^{-\mathrm{i}l\phi_n(u)}\, D_M\big(\phi_n(t)-\phi_n(u)\big)\,\sigma(u)dW(u)\right)^2 \nu(t)dt\right]$$

$$\leq C\; E\left[sup_{t\in[0,2\pi]}\nu^2(t)\right]\int_0^{2\pi}\int_0^t D_M^2\big(\phi_n(t)-\phi_n(u)\big)\,du\,dt \leq \frac{C}{M}$$

,

because of the hypotheses (**H.2**) and the properties of the discretized Dirichlet kernel proved in Clément and Gloter (2011). Using the same tools is possible to prove that the L_2-norm of (21) and (23) is

$$E\left[\left(H^1_{M,n}\right)^2\right] \leq \frac{C}{M^{\frac{2}{p}}} \text{ and } E\left[\left(H^3_{M,n}\right)^2\right] \leq \frac{C}{M^{\frac{2}{p}}}$$

for a $p \in (1,2)$.

By defining

$$\Gamma(t) = \int_0^t \mathrm{e}^{-\mathrm{i}l\phi_n(u)}\, D_M\big(\phi_n(t)-\phi_n(u)\big)\,\sigma(u)dW(u),$$

we can rewrite the term (22) in L_2-norm as

$$E\left[\left(H^2_{M,n}\right)^2\right] = \iint_0^{2\pi} E\left[\Gamma(t)\overline{\Gamma(t')}\,a(t)a(t')\right]\,dt\,dt'.$$

Using the duality for the stochastic integrals and the formula (1.65) in Nualart (2006), we get that

$$E\left[\Gamma(t)\overline{\Gamma(t')}a(t)a(t')\right] = E\left[\int_0^t \mathrm{e}^{-\mathrm{i}l\phi_n(u)} D_M\big(\phi_n(t)-\phi_n(u)\big)\sigma(u)\,\mathcal{D}_u\left(\overline{\Gamma(t')}\,a(t)a(t')\right)du\right]$$

$$= E\left[a(t)a(t')\int_0^t D_M\big(\phi_n(t)-\phi_n(u)\big)\,D_M\big(\phi_n(t')-\phi_n(u)\big)\,1_{\{u\leq t'\}}\,\nu(u)du\right]$$

$$+E\left[a(t)a(t')\int_0^t \mathrm{e}^{-\mathrm{i}l\phi_n(u)} D_M\big(\phi_n(t)\right.$$

$$\left.-\phi_n(u)\big)\left(\int_u^{t'} \mathrm{e}^{-\mathrm{i}l\phi_n(\mathrm{v})} D_M\big(\phi_n(t')-\phi_n(\mathrm{v})\big)\;\mathcal{D}_u(\sigma(\mathrm{v}))\,dW(\mathrm{v})\right)du\right]$$

$$+E\left[\overline{\Gamma(t')}\int_0^t \mathrm{e}^{-\mathrm{i}l\phi_n(u)} D_M\big(\phi_n(t)-\phi_n(u)\big)\,\sigma(u)\;\mathcal{D}_u\big(a(t)a(t')\big)\,du\right].$$

Then, we can consider

$$E\left[\left(H^2_{M,n}\right)^2\right] = E^1_{M,n} + E^2_{M,n} + E^3_{M,n}$$

.

Let us consider $p \in (1,2)$ in what follows. Using the Fubini's theorem

$$\begin{aligned} E^1_{M,n} &\le C \iint_0^{2\pi} \int_0^t \left| D_M\big(\phi_n(t)-\phi_n(u)\big)\, D_M\big(\phi_n(t')-\phi_n(u)\big) \right| 1_{\{u\le t'\}}\, du\, dt\, dt' \\ &= C \int_0^{2\pi} \left(\int_u^{2\pi} \left| D_M\big(\phi_n(t)-\phi_n(u)\big)\right| dt \int_u^{2\pi} \left| D_M\big(\phi_n(t')-\phi_n(u)\big)\right| dt' \right) du \\ &\le C\, sup_{u\in[0,2\pi]} \left(\int_0^{2\pi} \left| D_M\big(\phi_n(t)-\phi_n(u)\big)\right|^p dt \right)^{\frac{2}{p}} \le \frac{C}{M^{\frac{2}{p}}} \end{aligned}$$

where the properties of the Dirichlet kernel and the hypothesis (**H.2**) allow to get the estimation. We need the hypotheses (**H.2**) and (**H.3**) and the Cauchy-Schwartz inequality to estimate the addend $E^2_{M,n}$

$$\begin{aligned} E^2_{M,n} &\le C \iint_0^{2\pi} \left| \int_0^t \mathrm{e}^{-\mathrm{il}\phi_n(u)} D_M\big(\phi_n(t) \right. \\ &\qquad \left. -\phi_n(u)\big)\; E\left[\int_u^{t'} D_M^2\big(\phi_n(t')-\phi_n(\mathrm{v})\big)\, \mathcal{D}_u\big(\sigma(\mathrm{v})\big)^2\, d\mathrm{v}\right]^{\frac{1}{2}} du \right| dt\, dt' \\ &\le C\, E\left[sup_{u,\mathrm{v}\in[0,2\pi]}\, \mathcal{D}_u\big(\sigma(\mathrm{v})\big)^2\right]^{\frac{1}{2}} \iint_0^{2\pi} \int_0^t \left| D_M\big(\phi_n(t)-\phi_n(u)\big)\right| du \left[\int_u^{t'} D_M^2\big(\phi_n(t') \right. \\ &\qquad \left. -\phi_n(\mathrm{v})\big)\, d\mathrm{v}\right]^{\frac{1}{2}} dt\, dt' \le \frac{C}{M^{\frac{2+p}{2p}}}\ . \end{aligned}$$

As above, using similar arguments the same estimation can be obtained for the addend $E^3_{M,n}$.

$$\begin{aligned} E^3_{M,n} &\le C \iint_0^{2\pi} E\left[\int_0^{t'} D_M^2\big(\phi_n(t')-\phi_n(u)\big)\, \nu(u) du\right]^{\frac{1}{2}} \int_0^t \left| D_M\big(\phi_n(t)-\phi_n(u)\big)\right| du\, dt\, dt' \\ &\le C \iint_0^{2\pi} \left(\int_0^{t'} D_M^2\big(\phi_n(t')-\phi_n(u)\big)\, du \right)^{\frac{1}{2}} \left(\int_0^t \left| D_M\big(\phi_n(t)-\phi_n(u)\big)\right|^p du \right)^{\frac{1}{p}} dt\, dt' \le \frac{C}{M^{\frac{2+p}{2p}}} \end{aligned}$$

by means of the Cauchy-Schwartz inequality and the properties of the discretized Dirichlet kernel. Concerning the symmetric terms $\tilde{I}_{M,n}$, $\widetilde{H}^1_{M.n}$, $\widetilde{H}^2_{M,n}$ and $\widetilde{H}^3_{M,n}$respectively the same estimations can be carried out.

Therefore, the Fourier coefficients $c_l(\nu_{n,M})$ are bounded in L_2-norm, the difference

$$E\left[\left(c_l(\nu_{n,M})-c_l(\nu)\right)^2\right] \le C\,E\left[\left(\int_0^{2\pi}\left(\mathrm{e}^{-\mathrm{il}\phi_n(u)}-\ \mathrm{e}^{-\mathrm{ilu}}\right)\,\nu(t)\,dt\right)^2\right]+\frac{C}{M} \le C\,N^2\,\rho^2(n)+\,\frac{C}{M}$$

and the L_1-norm of (16) is an $\mathrm{O}\left(C\,N^2\,\rho^2(n)+\,\frac{C}{\sqrt{M}}\right)$ that goes to zero as $N, M, n\ \to\infty$ and $\rho(n)\to 0$ under the hypotheses (13).

It remains to study the convergence of the L_1-norm of the addend (17) to conclude the proof.
For any integer $|h|\le N$

$$\frac{2\pi}{2N+1}\sum_{|l|\le N}\mathrm{il}\,c_l\,(\nu)\,c_{h-l}(dp)\ -\ \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{iht}}\,\eta(t)\ dt$$
$$=\frac{2\pi}{2N+1}\sum_{|l|\le N}\left(\,c_l\,(d\nu)-\ c_0\,(d\nu)\right)c_{h-l}(dp)\ -\ \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{iht}}\,\eta(t)\ dt \tag{24}$$

because of the relation (8). Applying the Itô formula to the products $c_l\,(d\nu)c_{h-l}(dp)$ and $c_0\,(d\nu)c_{h-l}(dp)$, then (24) becomes in L_1-norm

$$\begin{aligned}E\Bigg[\Bigg|&\frac{1}{2\pi}\int_0^{2\pi}\int_0^{s}\mathrm{e}^{-\mathrm{ihu}}\,D_N(s-u)dp(u)d\nu(s)\\
&+\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{ihs}}\int_0^{s}D_N(s-u)\,d\nu(u)dp(s)\\
&-\frac{1}{2\pi}\int_0^{2\pi}\int_0^{s}\mathrm{e}^{-\mathrm{ihu}}D_N(u)\,dp(u)d\nu(s)\\
&-\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{ihs}}\int_0^{s}D_N(s)\,d\nu(u)dp(s)-\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{ihu}}D_N(u)\,\eta(u)du\ \Bigg|\Bigg]\\
&\le\frac{C}{\sqrt{N}}+\frac{C}{N^{\frac{1}{p}}}\end{aligned}$$

being p an integer greater than one. The above estimation is obtained under the hypotheses (**H**) and using the classical properties of the continuous Dirichlet kernel. Therefore we can conclude that also the addend (17) converges to zero in L_1-norm as $N\to\infty$ that concludes the proof.

Q.E.D

We conclude the section studying the consistency of the *spot* and the *integrated leverage estimators.*
For all $t\in(0,2\pi)$

$$\eta_{n,M,N}(t)=\sum_{|h|\le N}\left(1-\frac{|h|}{N}\right)\mathrm{e}^{\mathrm{iht}}\,\hat{c}_h\left(\eta_{n,M,N}\right) \tag{25}$$

The random function $\eta_{n,M,N}(t)$ will be called the *Fourier spot estimator* of the leverage function $\eta(t)$. The defined spot estimator uses all the information along the observed path to infer the value of $\eta_{n,M,N}(t)$ and by means of the Cesaro summation it allows to preserve the sign of the estimated function (see Remark 2.3 in Malliavin and Mancino (2009)).
An estimation of the integrated quantity (3) can be simply obtained by means of the definition (12) for h=0

$$\eta^{[1]}_{n,M,N}(t) = 2\pi\, \hat{c}_0\big(\eta_{n,M,N}\big). \tag{26}$$

Theorem 3.3. We assume that the hypotheses (**H**) and

$$\frac{N^2}{M} \to 0 \ \text{ and } \ M\rho(n) \to a$$

with $a \in \left(0, \frac{1}{2}\right)$, hold true as $n, N, M \to \infty$ and $\rho(n) \to 0$. Then we have the following convergence in probability

$$\eta^{[1]}_{n,M,N} \to \eta^{1}, \tag{27}$$

$$\lim_{n,N,M\to\infty} sup_{t\in(0,2\pi)} \left|\eta_{n,M,N}(t) - \eta(t)\right| = 0. \tag{28}$$

Proof:

In Theorem 3.1 we have proved that for any fixed *h* the convergence in probability of $\hat{c}_h\big(\eta_{n,M,N}\big)$ to the Fourier coefficient $c_h(\eta)$ as $n, N, M \to \infty$. Then, proving the convergence in (27) is straightforward and the uniform convergence in (28) follows by the Féjer Theorem for the continuous function.

Q.E.D

Remark 3.4. The extension of the estimation of the leverage in a multi-assets scenario is essentially contained in the proposed theory. Following the procedure described in Malliavin and Mancino (2002) for the estimation of the multivariate integrated and spot volatility it is possible to generalize this issue to the leverage estimation. We do not develop this theory in the present work. Nevertheless, the availability of a multivariate extension is an important feature of the estimators (25) and (26).

5 Conclusions

We have proposed estimators of the stochastic leverage function based on the use of the Fourier transform. The methodology used is non parametric and model free and rely only on a pre-estimation of the Fourier coefficients of the volatility function.
We obtain consistent estimators- *integrated and spot*- that show robustness in the presence of microstructure noise and irregular trading without any manipulation of the data. This is made possible by means of the choice of an appropriate number of Fourier coefficients of the return and of the volatility process to be included in the estimation.

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